Models using optimal mass transport for inverse design

Nonimaging Optics Minisymposium, 8 December 2025

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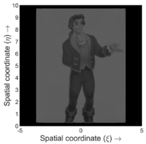
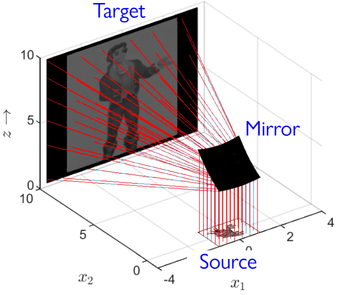
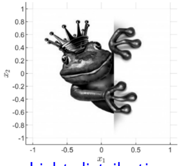
EINDHOVEN UNIVERSITY OF TECHNOLOGY

Can we turn a frog into a prince?



Light distribution target





Light distribution source

- *"In the original Grimm version of the story, the frog's spell was broken when the princess threw the frog against the wall, at which he transformed back into a prince, while in modern versions the transformation is triggered by the princess kissing the frog."*

  https://en.wikipedia.org/wiki/The_Frog_Prince

- Can we do it with light?

Romijn, L. B. (2021)
Generated Jacobian Equations in Freeform Optical Design: Mathematical Theory and Numerics
PhD thesis, Eindhoven University of Technology

Outline

Computational illumination optics at TU/e

Nonimaging freeform optics

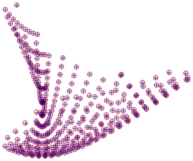
One mathematical framework for basic systems

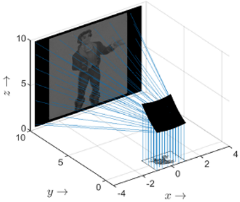
Iterative least-squares solver for 3D systems

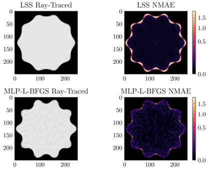
Numerical results

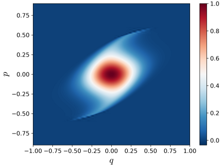
Beyond the basic systems

Conclusions and future research









Computational Illumination Optics Group at TU/e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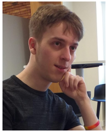
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Fatéma Goulamaly

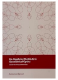

Roel Hacking

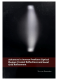

Willem Jansen


René Köhle


Sarah Nysen


Sanjana Verma


2025
Antonio Barion


2024
Teun van Roosmalen

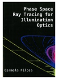

2024
Vì Kronberg


2023
Maikel Bertens


2023
Robert van Gestel

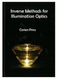

2021
Lotte Romijn


2018
Nitin Yadav



2018
Carmela Filosa


2017
Bart van Lith

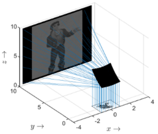


2014
Corien Prins

TU/e

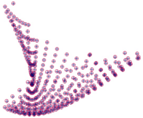
Research lines

Inverse design in nonimaging optics



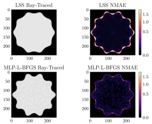
- Luminaires, street lights, …
- Compute freeform optical surfaces that convert given source into desired target distribution
- **Optimal transport, nonlinear PDE of Monge-Ampère type**

Inverse design in imaging optics



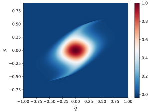
- Cameras, telescopes, …
- Make a very precise image of an object, minimizing aberrations
- **Hamiltonian optics, Lie transformations**

Optimization and generative AI for optical design



- Automatically design optical systems
- Handle constraints, uncertainties
- **Generative AI, multiobjective optimization**

Improved direct methods

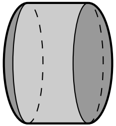


- Fast simulation of systems
- **Phase-space ray tracing, Liouville's equation**

Nonimaging freeform optics

Design of optical systems for illumination purposes

Optical system

Light source →  → Desired light distribution

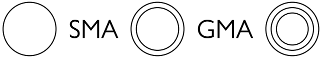
Industry standard: ray tracing

- Easy to implement
- Slow convergence
- Manual adjustments

Inverse methods

- Directly compute required optical system
- Need solving PDE of Monge-Ampère type
- Avoid iterations and manual optimization

Sixteen basic systems



Parallel target | Point target | Near-field target | Far-field target

Reflector systems

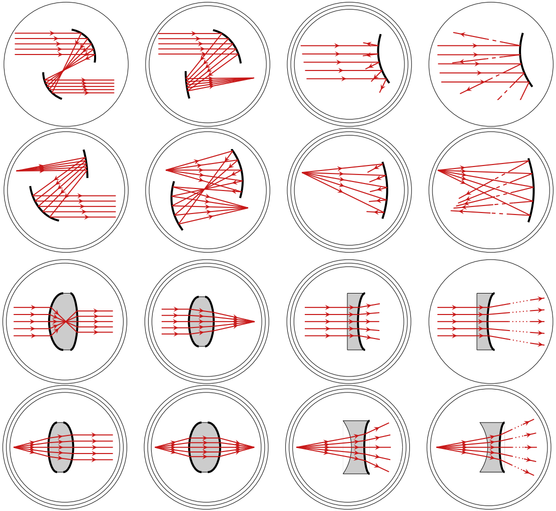
Parallel source

Point source

Lens systems

Parallel source

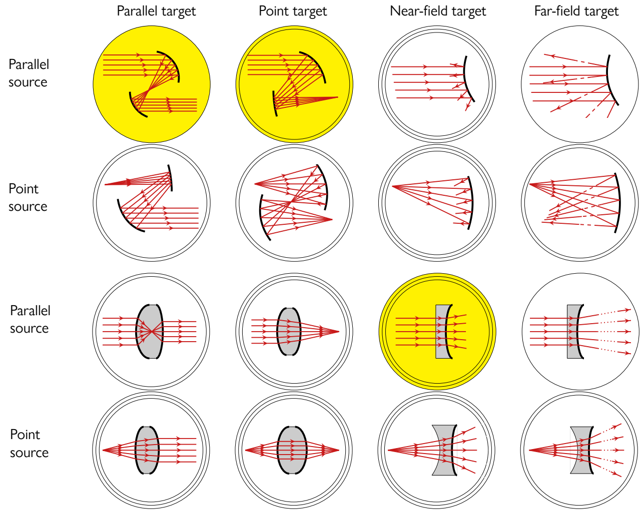
Point source



Sixteen basic systems

Reflector systems

Lens systems



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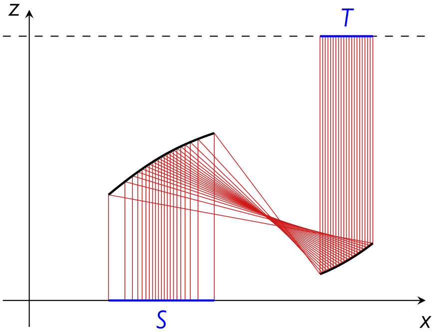
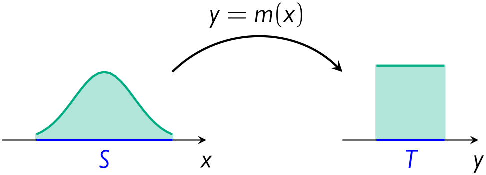
Parallel-to-parallel reflector 2D



- Source: parallel beam with light distribution $f(x)$, $x \in S$
- Target: parallel beam with light distribution $g(y)$, $y \in T$
- **Goal:** Find the two freeform reflector surfaces

- For the figure we used



Mathematical model

SMA

- Path of a ray
  - Leaves source S at $P = (x, 0)$
  - Hits first reflector at $A = (x, u(x))$
  - Hits second reflector at $B = (y, L - w(y))$
  - Arrives at target T at $Q = (y, L)$
- **Optical path length**

$$V = u(x) + d(A, B) + w(y)$$

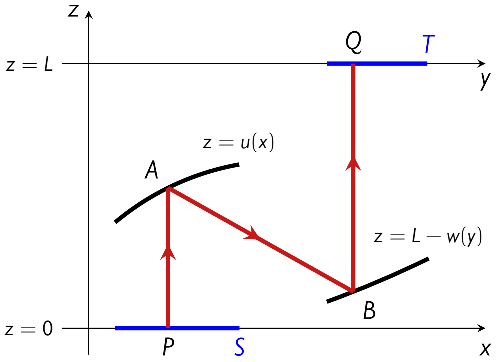
where $d(A, B)^2 = (y - x)^2 + (L - w(y) - u(x))^2$

- Eliminate $d(A, B)$ and rewrite as

$$u(x) + w(y) = \tfrac{1}{2}(V - L) + L - \frac{(y - x)^2}{2(V - L)}$$

$$=: c(x, y)$$

$c(x, y)$ is called cost function
Here: c is quadratic function



Optimal transport formulation

- Cost function relation

$$u(x) + w(y) = c(x, y)$$

Many solution pairs $(u(x), w(y))$

- Special choice: c-convex pair

$$u(x) = \max_{y \in T} \left(c(x, y) - w(y) \right)$$

$$w(y) = \max_{x \in S} \left(c(x, y) - u(x) \right)$$

- Necessary condition:

$$\frac{\partial c}{\partial x}(x, y) - u'(x) = 0$$

- For the current optical system, we have

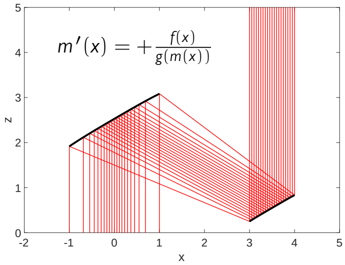
$$c(x, y) = \tfrac{1}{2}(V - L) + L - \frac{(y - x)^2}{2(V - L)}$$

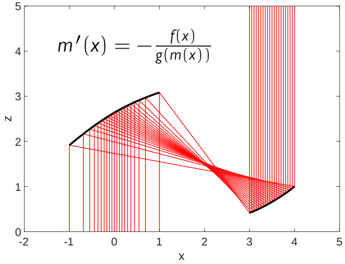
so

$$y = m(x) = x + (V - L)u'(x)$$

This is called the **optical mapping** $y = m(x)$

Energy conservation





- Source light distribution: $f = f(x)$, $x \in S$
  Target light distribution: $g = g(y)$, $y \in T$
  In the figure: $S = (-1, 1)$, $T = (3, 4)$
- Energy conservation:

$$\int_{-1}^{x} f(\xi)\, d\xi = \pm \int_{m(-1)}^{m(x)} g(y)\, dy = \pm \int_{-1}^{x} g(m(\xi))\, m'(\xi)\, d\xi$$

- Differentiate to x:

$$m'(x) = \pm \frac{f(x)}{g(m(x))}$$

  Solve ODE for m
- Differentiate $u(x) + w(y) = c(x, y)$ to x, substitute $y = m(x)$:

$$u'(x) = \frac{\partial c}{\partial x}(x, m(x))$$

  Solve ODE for u
- Second reflector: $w(m(x)) = c(x, m(x)) - u(x)$

Parallel-to-parallel reflector, 2D and 3D

SMA

2D

- Optical mapping:

$$y = m(x) = x + (V - L)u'(x) = \phi'(x)$$

- Optimal transport formulation:

$$u(x) + w(y) = c(x, y)$$
$$c(x, y) = \tfrac{1}{2}(V - L) + L - \frac{(y - x)^2}{2(V - L)}$$

- Energy conservation:

$$m'(x) = \phi''(x) = \pm \frac{f(x)}{g(m(x))}$$

3D: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$

- Optical mapping:

$$\mathbf{y} = \mathbf{m}(\mathbf{x}) = \mathbf{x} + (V - L)\nabla u(\mathbf{x}) = \nabla \phi(\mathbf{x})$$

- Optimal transport formulation:

$$u(\mathbf{x}) + w(\mathbf{y}) = c(\mathbf{x}, \mathbf{y})$$
$$c(\mathbf{x}, \mathbf{y}) = \tfrac{1}{2}(V - L) + L - \frac{\|\mathbf{y} - \mathbf{x}\|^2}{2(V - L)}$$

- Energy conservation:

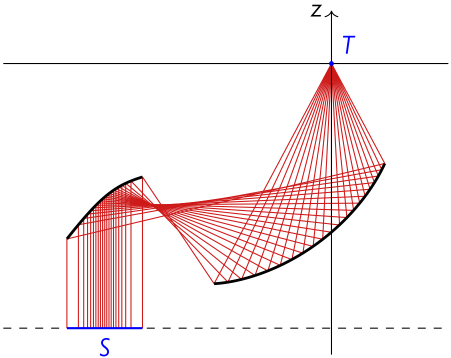
$$\det(D\mathbf{m}(\mathbf{x})) = \frac{\partial^2\phi}{\partial x_1^2}\frac{\partial^2\phi}{\partial x_2^2} - \left(\frac{\partial^2\phi}{\partial x_1\, \partial x_2}\right)^2 = \pm \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

Standard Monge-Ampère equation

Second-order nonlinear PDE

Parallel-to-point reflector 2D

GMA



- Source: parallel beam with light distribution $f(x)$
- Target: point with light distribution $g(y)$
- **Goal:** Find the two freeform reflector surfaces

TU/e

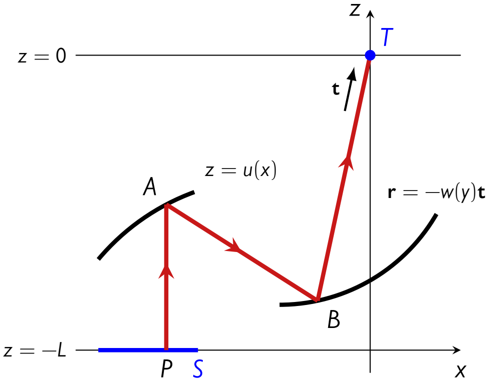
Mathematical model

GMA

- Path of a ray
  - Leaves source S at $P = (x, -L)$
  - Hits first reflector at $A = (x, -L + u(x))$
  - Hits second reflector at $B = (-w(y)t_1, -w(y)t_2)$
  - Arrives at target $T = (0, 0)$
- Optical path length: $V = u(x) + d(A, B) + w(y)$
- Geometric relation ($\beta := V - L$)

$$\left(-\frac{u}{\beta} - \frac{x^2}{2\beta^2} + \frac{V+L}{2\beta}\right) \cdot \left(\frac{\beta}{w}(1+y^2) - 2y^2\right) = \left(1 + \frac{xy}{\beta}\right)^2$$

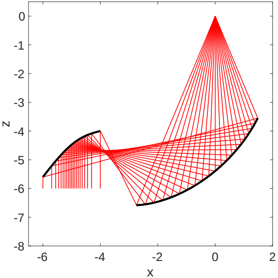
- Take logarithms: $u_1(x) + u_2(y) = c(x, y)$
  Cost function is non-quadratic

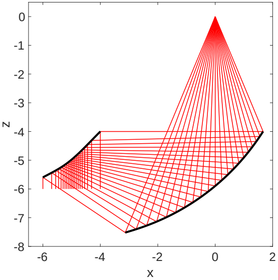


TU/e

Numerical method

GMA





- Energy conservation:

$$m'(x) = \pm \frac{f(x)}{g(m(x))} \frac{1 + (m(x))^2}{2}$$

Solve ODE for m

- Differentiate $u_1(x) + u_2(y) = c(x, y)$ to x, substitute $y = m(x)$:

$$u_1'(x) = \frac{\partial c}{\partial x}(x, m(x))$$

Solve ODE for u_1
From u_1 compute u

- Second reflector:

$$u_2(m(x)) = c(x, m(x)) - u_1(x)$$

From u_2 compute w

TU/e

Parallel-to-point reflector, 2D and 3D

GMA

2D

- Energy conservation:

$$m'(x) = \pm \frac{f(x)}{g(m(x))} \frac{1+(m(x))^2}{2}$$

- Optimal transport formulation:

$$u_1(x) + u_2(y) = c(x,y)$$

$$c(x,y) = \log\left(\left(1+\frac{xy}{\beta}\right)^2\right)$$

3D: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$

- Energy conservation:

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \pm \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))} \frac{\left(1+\|\mathbf{m}(\mathbf{x})\|^2\right)^2}{4}$$

- Optimal transport formulation:

$$u_1(\mathbf{x}) + u_2(\mathbf{y}) = c(\mathbf{x},\mathbf{y})$$

$$c(\mathbf{x},\mathbf{y}) = \log\left(\left(1+\frac{\mathbf{x}\cdot\mathbf{y}}{\beta}\right)^2\right)$$

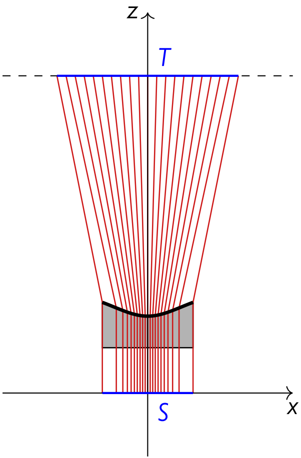
- Differentiate, substitute $\mathbf{y} = \mathbf{m}(\mathbf{x})$, differentiate

$$\underbrace{\mathrm{D}_{\mathbf{xy}}c(\mathbf{x},\mathbf{m}(\mathbf{x}))}_{\mathbf{C}} \, \mathrm{D}\mathbf{m}(\mathbf{x}) = \underbrace{\mathrm{D}^2 u_1(\mathbf{x}) - \mathrm{D}_{\mathbf{xx}}c(\mathbf{x},\mathbf{m}(\mathbf{x}))}_{\mathbf{P}}$$

- **Generalized Monge-Ampère equation**

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \frac{\det(\mathbf{P})}{\det(\mathbf{C})} = \pm \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))} \frac{\left(1+\|\mathbf{m}(\mathbf{x})\|^2\right)^2}{4}$$

Parallel-to-near-field lens



- Source: parallel beam with light distribution $f(x)$
- Target: near-field with light distribution $g(y)$
- **Goal:** Find the single freeform lens surface

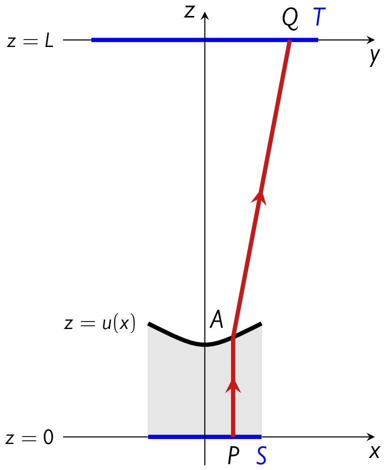
Mathematical model

GJE

- Path of a ray
  - Leaves source S at $P = (x, 0)$
  - Hits freeform lens surface at $A = (x, u(x))$
  - Arrives at target T at $Q = (y, L)$
- Optical path length:

$$
\begin{aligned}
V(y) &= n \cdot u(x) + d(A, Q) \\
&= n \cdot u(x) + \sqrt{(y - x)^2 + (L - u(x))^2}
\end{aligned}
$$

Cannot be formulated as optimal transport problem



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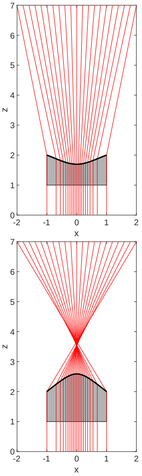
Numerical method

GJE



- Energy conservation:

$$m'(x) = \pm \frac{f(x)}{g(m(x))}$$

Solve ODE for m

- Geometric relation:

$$V(y) = n \cdot u(x) + \sqrt{(y-x)^2 + (L-u(x))^2}$$

differentiate to x, substitute $y = m(x)$, solve for $u'(x)$:

$$u'(x) = \frac{m(x) - x}{n \cdot \sqrt{(m(x)-x)^2 + (L-u(x))^2} + u(x) - L}$$

Solve ODE for u

TU/e

Parallel-to-near-field lens, 2D and 3D

GJE

2D

- Energy conservation:

$$m'(x) = \pm \frac{f(x)}{g(m(x))}$$

- Geometric relation:

$$V(y) = n \cdot u(x) + \sqrt{(y-x)^2 + (L - u(x))^2}$$

3D

- Energy conservation:

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \pm \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

- Geometric relation:

$$V(\mathbf{y}) = n \cdot u(\mathbf{x}) + \sqrt{\|\mathbf{y} - \mathbf{x}\|^2 + (L - u(\mathbf{x}))^2}$$
$$=: H(\mathbf{x}, \mathbf{y}, u(\mathbf{x}))$$

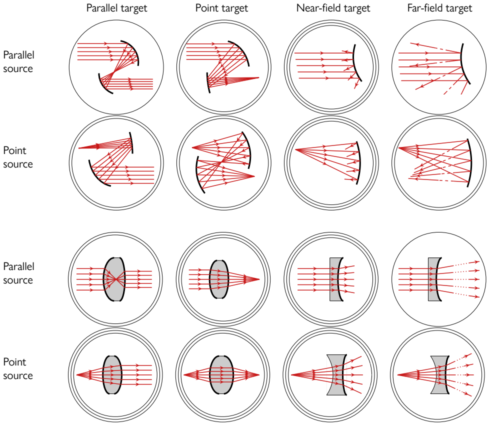
- Differentiate to $\mathbf{x}$: $\nabla_{\mathbf{x}} \big(H(\mathbf{x}, \mathbf{y}, u(\mathbf{x})) \big) = \mathbf{0}$
- Let $\widetilde{H}(\mathbf{x}, \mathbf{y}) := H(\mathbf{x}, \mathbf{y}, u(\mathbf{x}))$, substitute $\mathbf{y} = \mathbf{m}(\mathbf{x})$, differentiate to $\mathbf{x}$:

$$\mathrm{D}_{\mathbf{xx}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x})) + \underbrace{\mathrm{D}_{\mathbf{xy}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x}))}_{\mathbf{C}} (\mathrm{D}\mathbf{m})(\mathbf{x}) = \mathbf{O}$$

- **Generated Jacobian equation**

$$\det\big(\mathrm{D}_{\mathbf{xx}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x}))\big) = \pm \det(\mathbf{C}) \frac{f(\mathbf{x})}{g(\mathbf{m}(\mathbf{x}))}$$

Unified mathematical framework



- Hamilton characteristic functions
- Energy conservation: $\det(\mathrm{D}\mathbf{m}) = F(\mathbf{x}, \mathbf{m}(\mathbf{x}))$
- Matrix equation optical map: $\mathbf{P} = \mathbf{C}\,\mathrm{D}\mathbf{m}$
- Model 1: **Standard Monge-Ampère** (SMA)
  Optimal transport formulation, $c(\mathbf{x}, \mathbf{y})$ quadratic
  $$\mathbf{C} = \mathbf{I} \qquad \mathbf{P} = \mathrm{D}^2 u$$
- Model 2: **Generalized Monge-Ampère** (GMA)
  Optimal transport formulation, $c(\mathbf{x}, \mathbf{y})$ non-quadratic
  $$\mathbf{C} = \mathrm{D}_{\mathbf{xy}} c \qquad \mathbf{P} = \mathrm{D}^2 u_1 - \mathrm{D}_{\mathbf{xx}} c$$
- Model 3: **Generated Jacobian equation** (GJE)
  No optimal transport formulation
  $$\mathbf{C} = \mathrm{D}_{\mathbf{xy}} \widetilde{H} \qquad \mathbf{P} = -\mathrm{D}_{\mathbf{xx}} \widetilde{H}$$
- Least-squares solver

Anthonissen, M.J.H., Romijn, L.B., ten Thije Boonkkamp, J.H.M., IJzerman, W.L. (2021). *Unified mathematical framework for a class of fundamental freeform optical systems*, Optics Express, 29(20), 31650-31664, https://doi.org/10.1364/OE.438920

Romijn, L. B. (2021). *Generated Jacobian Equations in Freeform Optical Design: Mathematical Theory and Numerics*, PhD thesis, Eindhoven University of Technology

Iterative least-squares solver for 3D systems

SMA GMA GJE

- Find mapping $\mathbf{m} : S \to T$ such that

$$\det(\mathrm{D}\mathbf{m}) = F(\mathbf{x}, \mathbf{m}(\mathbf{x}))$$
$$\mathbf{m}(\partial S) = \partial T$$

- Break down in substeps
  We compute $\mathbf{P}$, $\mathbf{b}$, $\mathbf{m}$ such that

$$\mathbf{P} = \mathbf{C}\,\mathrm{D}\mathbf{m}$$
$$\det(\mathbf{P}) = \det(\mathbf{C})\,F(\mathbf{x}, \mathbf{m}(\mathbf{x}))$$
$$\mathbf{b}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) \qquad \mathbf{x} \in \partial S$$

  $\mathbf{b}$ maps ∂S to ∂T

- Find surface from $\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{m}(\mathbf{x})) = \nabla u_1(\mathbf{x})$

- Iterative procedure:
  1. Choose an initial guess $\mathbf{m}^0$
     Let n = 0
  2. Let $J_I(\mathbf{m}, \mathbf{P}) = \frac{1}{2} \iint_S \|\mathbf{C}\,\mathrm{D}\mathbf{m} - \mathbf{P}\|^2 \, d\mathbf{x}$

     $$\mathbf{P}^{n+1} = \underset{\mathbf{P}}{\operatorname{argmin}}\; J_I(\mathbf{m}^n, \mathbf{P})$$

     Constrained minimization problem
  3. Let $J_B(\mathbf{m}, \mathbf{b}) = \frac{1}{2} \int_{\partial S} \|\mathbf{m} - \mathbf{b}\|^2 \, ds$

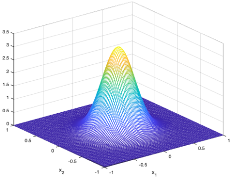
     $$\mathbf{b}^{n+1} = \underset{\mathbf{b}}{\operatorname{argmin}}\; J_B(\mathbf{m}^n, \mathbf{b})$$

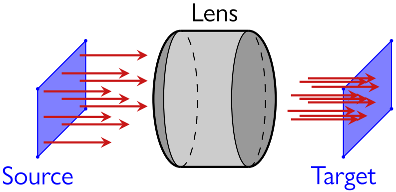
     Projection on boundary ∂T
  4. Let $J(\mathbf{m}, \mathbf{P}, \mathbf{b}) = \alpha J_I(\mathbf{m}, \mathbf{P}) + (1 - \alpha) J_B(\mathbf{m}, \mathbf{b})$

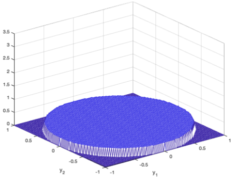
     $$\mathbf{m}^{n+1} = \underset{\mathbf{m}}{\operatorname{argmin}}\; J(\mathbf{m}, \mathbf{P}^{n+1}, \mathbf{b}^{n+1})$$

     Elliptic PDEs for m_1 and m_2 — FVM
  5. Let $n := n + 1$, go to Step 2

Laser beam shaping (parallel-to-parallel lens 3D)







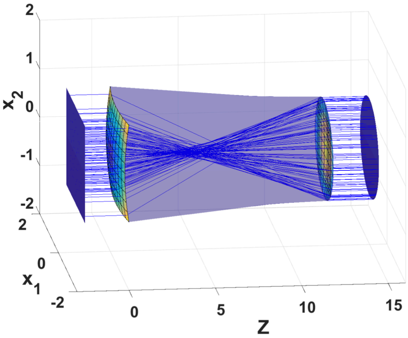
Source distribution:
Gaussian profile
Source emits parallel light rays

Optical system:
one lens with two
freeform surfaces

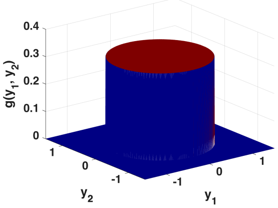
Desired target distribution:
circular top hat profile
Output beam: parallel light rays

Goal
Find the two freeform surfaces

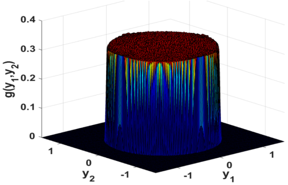
Numerical results for the laser beam shaping problem



Desired target distribution

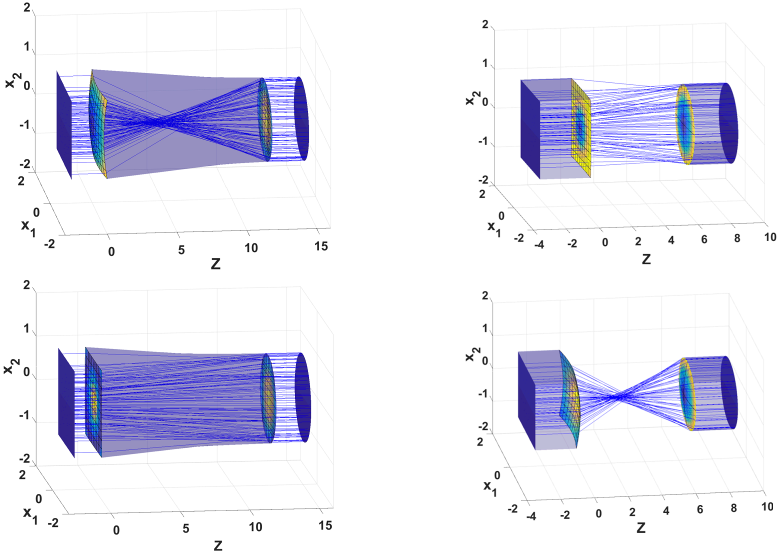


Achieved

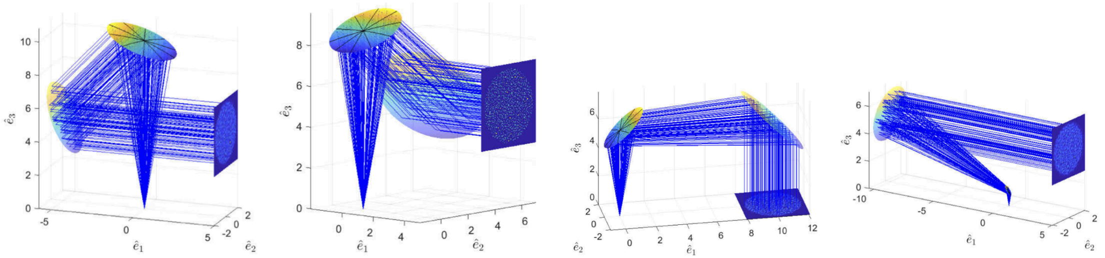


Yadav, N.K., ten Thije Boonkkamp, J.H.M., IJzerman, W.L. (2019)
Computation of double freeform optical surfaces using a Monge–Ampère solver: Application to beam shaping
Optics Communications, 439, 251-259
https://doi.org/10.1016/j.optcom.2019.01.069

Alternative optical systems

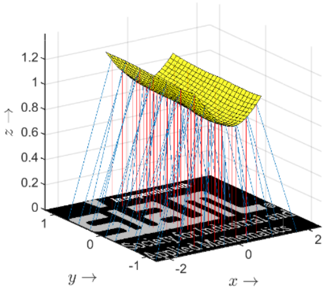


Point-to-parallel reflector system 3D

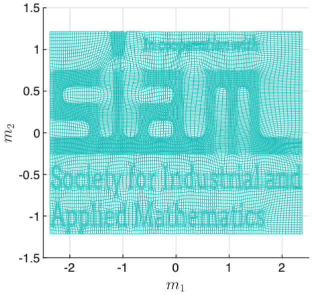


Van Roosmalen, A. H., Anthonissen, M. J. H., IJzerman, W. L., ten Thije Boonkkamp, J. H. M. (2021)
Design of a freeform two-reflector system to collimate and shape a point source distribution
Optics Express, 29(16), 25605-25625
https://doi.org/10.1364/OE.425289

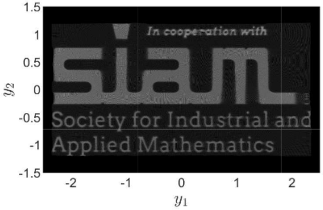
Parallel-to-near-field reflector 3D



Surface



Mapping

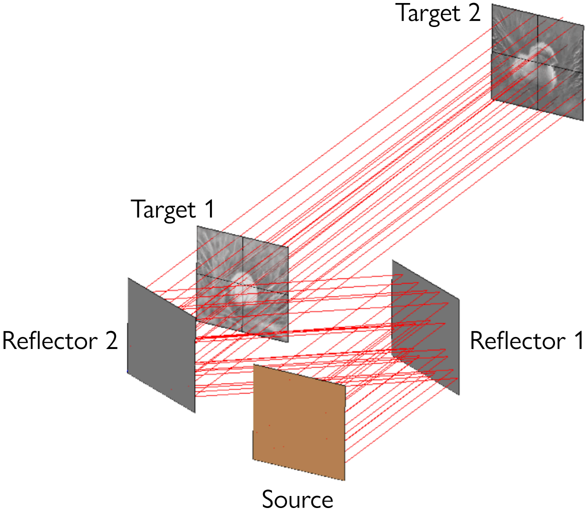


Ray trace

Romijn, L.B., Anthonissen, M.J.H., ten Thije Boonkkamp, J.H.M., IJzerman, W.L. (2021)
An iterative least-squares method for generated Jacobian equations in freeform optical design
SIAM Journal on Scientific Computing, 43(2), B298-B322
https://doi.org/10.1137/20M1338940

TU/e

The chicken or the egg?



Light distributions

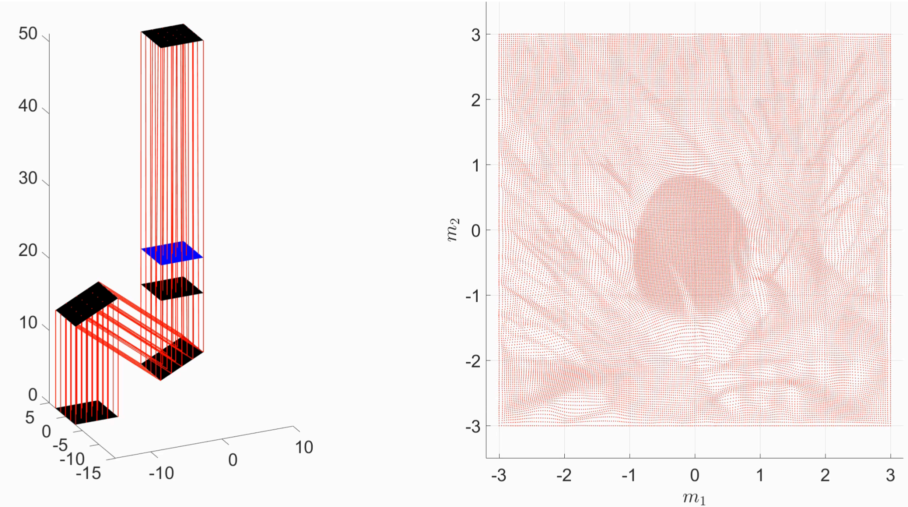
Source Target 1 Target 2

Goal

Find the freeform reflector surfaces

Braam, P.A., ten Thije Boonkkamp, J.H.M., Anthonissen, M.J.H., Mitra, K., Beltman, R., IJzerman, W.L. (2025)
Inverse freeform design of a parallel-to-two-target reflector system
Submitted

Animation for the two-target reflector system



This slide has an animation. It is available online at https://youtu.be/ZwXnc-QXyBs

TU/e

Conclusions and future research

Conclusions

- Mathematical models for freeform optical design
- One framework for basic systems
- Least-squares solver directly computes freeform optical surfaces
- Extension: parallel source to two illuminance patterns

Future research

- Use AI for optical design
- Model finite sources
- Model F-GRIN optics
- Include phase information

More info: https://martijna.win.tue.nl/Optics/

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